J Stat Phys (2010) 139: 800–819
DOI 10.1007/s10955-010-9960-5

Physical Measures for Multivalued Inverse Iterates Near Hyperbolic Repellors

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Received: 25 November 2009 / Accepted: 31 March 2010 / Published online: 21 April 2010

Abstract We study new invariant probability measures, describing the distribution of multivalued inverse iterates (i.e. of different local inverse iterates) for a non-invertible smooth function f which is hyperbolic, but not necessarily expanding on a repellor Λ. The methods for the higher dimensional non-expanding and non-invertible case are different than the ones for diffeomorphisms, due to the lack of a nice unstable foliation (local unstable manifolds depend on prehistories and may intersect each other, both in Λ and outside Λ), and the fact that Markov partitions may not exist on Λ. We obtain that for Lebesgue almost all points z in a neighbourhood V of Λ, the normalized averages of Dirac measures on the consecutive preimage sets of z converge weakly to an equilibrium measure μ^- on Λ; this implies that μ^- is a physical measure for the local inverse iterates of f. It turns out that μ^- is an inverse SRB measure in the sense that it is the only invariant measure satisfying a Pesin type formula for the negative Lyapunov exponents. Also we show that μ^- has absolutely continuous conditional measures on local stable manifolds, by using the above convergence of measures. We prove then that $f : (\Lambda, \mathcal{B}(\Lambda), \mu^-) \to (\Lambda, \mathcal{B}(\Lambda), \mu^-)$ cannot be one-sided Bernoulli, although it is an exact endomorphism of Lebesgue spaces. Several classes of examples of hyperbolic non-invertible and non-expanding repellors, with their inverse SRB measures, are given in the end.

Keywords Hyperbolic non-invertible maps (endomorphisms) · Repellors · SRB measures for endomorphisms · Physical and equilibrium measures · 1-sided Bernoulli maps

1 Introduction

SRB measures (Sinai, Ruelle, Bowen) and physical measures have been studied for many classes of dynamical systems having some form of hyperbolicity, either uniform, partial or non-uniform ([2, 8, 17, 20, 21], etc.). Intuitively physical measures describe the distributions

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of forward iterates in a neighbourhood of an attractor. SRB measures are usually defined by the absolute continuity of their conditional measures on local unstable manifolds ([21]). The term *physical measures* was introduced by Eckmann and Ruelle ([4]) who also proved many of their properties and gave relations to examples from physics (turbulence theory, statistical mechanics, strange attractors, etc.). Measure-theoretic entropy and Lyapunov exponents prove to be very important with regard to physical and SRB measures, as in Pesin's entropy formula ([4, 6, 7, 21], etc.). For uniformly hyperbolic diffeomorphisms having an attractor Λ, and for Anosov diffeomorphisms, physical measures are in fact SRB measures as was proved by Sinai, Ruelle, Bowen ([2, 8, 17, 20, 21]). For other systems there may exist physical measures which are not SRB (as in [4]). In [4], it was studied mainly the case of attractors for diffeomorphisms or the case of a flow indexed with both positive and negative parameters t. In such a case the inverse of the map is well defined and it is also a smooth map. For flows we simply can take $f^t, t < 0$. One cannot do the same if the dynamical system is not invertible.

In this paper we focus on finding physical measures giving the *distribution of consecutive preimage sets* for non-invertible smooth maps (such maps will be called *endomorphisms*), in the vicinity of a hyperbolic repellor. There are many examples of systems which are not invertible, for instance the non-invertible horseshoes from [1], s-hyperbolic holomorphic maps in several dimensions and their invariant sets ([9]), skew products having a finite iterated function system in the base and overlaps in their fibers, hyperbolic toral endomorphisms, examples with invariant folded drapes or veils ([4]), baker's transformations with overlaps, hyperbolic basic sets with locally constant number of preimages (see [13]), etc. By similarity to the SRB measure, one natural question would be to study the *distribution of various preimages near a hyperbolic repellor* Λ. The problem is that there is *no* unique inverse f^{-1}; instead, if f does not have any critical points near the Λ, we will obtain local inverse iterates, or equivalently a *multivalued inverse iterate* of f. If f is locally d-to-1 on a basic set Λ, and if the local inverse iterates of f on some open set W are denoted by $f^{-1}_{W,1}, \dots, f^{-1}_{W,d}$, then the multivalued inverse of f on W is $(f^{-1}_{W,1}, \dots, f^{-1}_{W,d})$. Knowing the behaviour of inverse trajectories of a system may be important when we want to obtain information about the past states of the system.

It is important to keep in mind that the map f *is not assumed expanding* on Λ; indeed for the expanding case a lot is known about the distribution of preimages (see [8, 19]) and the situation is characterized by the fact that local inverse iterates decrease exponentially fast the diameter of small balls; this guarantees that we have bounded distortion lemmas. However in the general higher dimensional non-invertible hyperbolic case we do not have control on the distortion of small balls under local inverse iterates; indeed they may increase in the stable direction in backward time.

Non-invertibility brings many difficulties into the setting, like not being able to apply directly Birkhoff Ergodic Theorem for f^{-1} like in the case of diffeomorphisms, the non-existence of a Markov partition of Λ (as f is just an endomorphism, not necessarily expanding on Λ), etc. One classical tool when dealing with endomorphisms would be to use the natural extension $\hat{\Lambda}$ of Λ (also known as the inverse limit), but then one looses differentiability properties near Λ, as $\hat{\Lambda}$ is not a manifold. In general for endomorphisms, local unstable manifolds depend on whole prehistories not only on the base points ([18]); this dependence is Holder continuous with respect to prehistories ([10]). Our repellors will be in fact unions of global stable sets, but the overlappings and foldings of the system introduce a complicated and very irregular dynamics. Moreover the number of preimages belonging to Λ of a given point may vary a priori along Λ.

For attractors/repellors Λ for diffeomorphisms f we know that there exists an SRB/inverse SRB measure on Λ and that $(\Lambda, f|_\Lambda)$ becomes a Bernoulli 2-sided transformation

([2, 8]). This is based mainly on the existence of Markov partitions in the invertible case ([2, 20]). Also for expanding maps there exist Markov partitions ([17, 19]) and the system is isomorphic to a 1-sided Markov chain. In the non-invertible non-expanding case we however do not have Markov partitions, as mentioned above. We will show that in our non-invertible case, if Λ is a repellor with its inverse SRB measure μ^- then $(\Lambda, f|_\Lambda, \mu^-)$ *is not 1-sided Bernoulli*. This is in clear contrast with what happens for diffeomorphisms or expanding maps.

The *main directions and results* of the paper are the following:

First we will specify what we understand by a repellor, in Definition 1. We prove that on a repellor Λ, the number of preimages belonging to Λ of any $x \in \Lambda$ is locally constant. We also show a very important property of these sets, namely the stability under perturbations, in Proposition 3. Then we prove in Theorem 1 that the pressure of the stable potential Φ^s along a connected repellor Λ is related to the number d of preimages of an arbitrary point, which remain in Λ.

We will define next the probability measures

$$\mu_n^z := \frac{1}{d^n} \sum_{y \in f^{-n}z \cap U} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i y}, \quad n \geq 1, z \in V \subset U,$$

where V, U are close enough neighbourhoods of Λ. In Theorem 2 we give the main result, namely the weak convergence of the measures μ_n^z towards the equilibrium measure μ_s of the potential Φ^s, for Lebesgue almost all points $z \in V$. In this Theorem, Λ will be assumed connected (not a very restrictive assumption for our notion of repellor, as will be seen). We show then in Theorem 3 that a Pesin type formula involving the *negative Lyapunov exponents* can be derived for this physical measure $\mu^- = \mu_s$. This will give also the absolute continuity of conditional measures of μ^- on stable manifolds, by using the convergence of measures of Theorem 2 and a result of Liu ([7]) relating entropy, folding entropy and negative Lyapunov exponents. In fact by using the convergence of the measures $(\mu_n^z)_n$ from Theorem 2, we show that the folding entropy $H_{\mu^-}(\epsilon/f^{-1}\epsilon)$ is equal to $\log d$, where ϵ is the partition of Λ into single points. Therefore by all these properties, it follows that μ^- can be viewed as an *inverse SRB measure*.

The above inverse Pesin type formula will imply in Theorem 4 that the repellor Λ with its inverse SRB measure μ^- is not isomorphic to a one-sided Bernoulli shift. This is in contrast with the case of attractors for diffeomorphisms where the attractor, together with its SRB measure, is 2-sided Bernoulli. We show however in Theorem 5 that μ^- has Exponential Decay of Correlations on Holder potentials; and that $f|_\Lambda : (\Lambda, \mu^-) \to (\Lambda, \mu^-)$ is *exact* as an endomorphism of Lebesgue spaces, hence mixing of any order ([16]).

The problem of isomorphisms to 1-sided Bernoulli shifts is delicate for smooth constant-to-one endomorphisms, and is fundamentally different than the one in the case of diffeomorphisms and 2-sided Bernoulli shifts. In [3], Bruin and Hawkins gave criteria and several classes of one-dimensional real or complex maps and measures, which are not 1-sided Bernoulli. By contrast to the case of Bernoulli automorphisms, measure-theoretic entropy is *not* a complete invariant for 1-sided Bernoulli shifts. The connected repellors from our paper represent new examples of invariant sets on which smooth endomorphisms are constant-to-one. We find thus new classes of smooth endomorphisms and natural invariant measures on basic sets (not necessarily expanding), which are *not 1-sided Bernoulli*. Still, they will be shown to display the strong mixing properties mentioned above (Exponential Decay of Correlations, and mixing of any order).

Finally we describe some classes of examples in Sect. 3, among which hyperbolic toral endomorphisms, other Anosov endomorphisms, as well as new classes of non-expanding repellors which are not Anosov, together with their inverse SRB measures.

2 Main Results

First we will specify what do we understand by *repellor*. As a *general setting* throughout the paper, we consider $f : M \to M$ a smooth (say $\mathcal{C}^2$) map on a Riemannian manifold, and Λ an f-invariant compact set in M which does not intersect the critical set C_f of f. We remark that the preimages of a point from Λ do not have to remain in Λ necessarily. Also let us notice that if C_f would intersect Λ, the basic ideas would remain the same as long as we assume an integrability condition on $\log |Df_s|$ over Λ.

Definition 1 Let $f : M \to M$ be a smooth (for example $\mathcal{C}^2$) map on a Riemannian manifold and let Λ be a compact set which is f-invariant (i.e. $f(\Lambda) = \Lambda$) and s.t $f|_\Lambda$ is topologically transitive; assume also that there exists a neighbourhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U$. Such a set will be called a *basic set* for f ([5]). We say that Λ is a *repellor* for f if Λ is a basic set for f, $C_f \cap \Lambda = \emptyset$ and if there exists a neighbourhood U of Λ such that $\bar{U} \subset f(U)$.

We will call any point $y \in f^{-1}(x)$ an f-*preimage* of $x \in M$; and by n-*preimage* of x we mean any point $y \in f^{-n}(x)$, for an integer $n > 0$.

Proposition 1 *In the setting of Definition 1, if Λ is a repellor for f, then $f^{-1}\Lambda \cap U = \Lambda$. If moreover Λ is assumed to be connected, the number of f-preimages that a point has in Λ is constant.*

Proof Let a point $x \in \Lambda$, and y be an f-preimage of x from U. Then $f^n y \in \Lambda$, $n \geq 1$. From Definition 1, since Λ is assumed to be a repellor, the point y has a preimage y_{-1} in U; then y_{-1} has a preimage y_{-2} from U, and so on. Thus y has a full prehistory belonging to U and also its forward orbit belongs to U, hence $y \in \Lambda$ since Λ is a basic set. So $f^{-1}\Lambda \cap U = \Lambda$.

We prove now the second part of the statement. Let a point $x \in \Lambda$ and assume that it has d f-preimages in Λ, denoted $x_1, \ldots, x_d$. Consider also another point $y \in \Lambda$ close to x. If y is close enough to x and since $\mathcal{C}_f \cap \Lambda = \emptyset$, it follows that y also has exactly d f-preimages in U, denoted by $y_1, \ldots, y_d$. Since from the first part we know that $f^{-1}\Lambda \cap U = \Lambda$, we obtain that $y_1, \ldots, y_d \in \Lambda$. In conclusion the number of f-preimages in Λ of a point is locally constant. If Λ is assumed to be connected, then the number of preimages belonging to Λ of any point from Λ, must be constant. □

Let us denote by $d(x)$ the number of f-preimages that the point x has in the repellor Λ. Then from the above Proposition we know that $d(\cdot)$ is locally constant on Λ. Clearly there exist only finitely many values that $d(\cdot)$ may take on Λ. We will assume in the sequel that the number of preimages $d(\cdot)$ is *constant* on Λ. This happens for instance when Λ is connected (from Proposition 1). We give the results in this setting (i.e. when Λ is connected), but in fact all we need is that $d(\cdot)$ is constant.

We will work with uniformly hyperbolic endomorphisms on Λ ([1, 10, 11, 18], etc.) The stable tangent spaces $E^s_x, x \in \Lambda$ depend Holder continuously on x (see [5, 10, 12]); the unstable tangent spaces depend on whole prehistories, i.e. we have $E^u_{\hat{x}}, \hat{x} \in \hat{\Lambda}$. Here $(\hat{\Lambda}, \hat{f})$ is the *natural extension* ([15]), or *inverse limit* of the dynamical system (Λ, f);

the space $\hat{\Lambda} := \{\hat{x} = (x, x_{-1}, x_{-2}, \ldots), f(x_{-i}) = x_{-i+1}, i \geq 1, x_0 := x\}$ is the space of full prehistories of points from Λ and the map $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$, $\hat{f}(\hat{x}) = (fx, x, x_{-1}, x_{-2}, \ldots)$, $\hat{x} \in \hat{\Lambda}$ is the *shift homeomorphism*. We denote also by $\pi : \hat{\Lambda} \to \Lambda$ the *canonical projection* given by $\pi(\hat{x}) = x$, $\hat{x} \in \hat{\Lambda}$. The compact topological space $\hat{\Lambda}$ can be endowed with a natural metric, but it is not a manifold.

We shall denote $Df|_{E^s_x}$ by $Df_s(x)$ and call it the *stable derivative* at $x \in \Lambda$; and $Df_u(\hat{x}) := Df|_{E^u_{\hat{x}}}$ is the *unstable derivative* at $\hat{x} \in \hat{\Lambda}$. Similarly the local stable and unstable manifolds are denoted by $W^s_r(x)$, $W^u_r(\hat{x})$, $\hat{x} \in \hat{\Lambda}$, for some small $r > 0$. We call *stable potential* the function

$$\Phi^s(x) := \log |Jac(Df_s(x))| = \log |\det(Df_s(x))|, \quad x \in \Lambda.$$

One notices that there exists a bijection between the set $\mathcal{M}(f)$ of f-invariant probability measures on Λ and the set $\mathcal{M}(\hat{f})$ of $\hat{f}$-invariant probability measures on the natural extension $\hat{\Lambda}$, so that to any measure $\mu \in \mathcal{M}(f)$ we associate the unique measure $\hat{\mu} \in \mathcal{M}(\hat{f})$ satisfying the relation $\pi_*(\hat{\mu}) = \mu$ (for example Rokhlin, [15]). It is easy to show that $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$ and that $P_{\hat{f}}(\phi \circ \pi) = P_f(\phi)$, $\forall \phi \in \mathcal{C}(\Lambda, \mathbb{R})$. Thus μ is an equilibrium measure for a potential ϕ if and only if its unique $\hat{f}$-invariant lifting $\hat{\mu}$ is an equilibrium measure for $\phi \circ \pi$ on $\hat{\Lambda}$. Next let us transpose to the setting of endomorphisms, some properties of equilibrium measures from the diffeomorphism case, by using liftings to the natural extension.

In the sequel, given $y \in \Lambda, n \geq 1$ and $\varepsilon > 0$, we denote by $B_n(y, \varepsilon) := \{z \in M, d(f^i z, f^i y) < \varepsilon, i = 0, \ldots, n-1\}$ a Bowen ball. For a continuous real function ϕ (defined on the f-invariant set Λ) and for a positive integer n, we define the "consecutive sum" by:

$$S_n\phi(y) := \phi(y) + \phi(f(y)) + \cdots + \phi(f^{n-1}(y)), \quad y \in \Lambda.$$

Proposition 2 *Let Λ be a hyperbolic basic set for a smooth endomorphism $f : M \to M$, and let ϕ a Holder continuous function on Λ. Then there exists a unique equilibrium measure μ_ϕ for ϕ on Λ such that for any $\varepsilon > 0$, there exist positive constants A_ε, B_ε so that for any $y \in \Lambda, n \geq 1$,*

$$A_\varepsilon e^{S_n\phi(y) - nP(\phi)} \leq \mu_\phi(B_n(y, \varepsilon)) \leq B_\varepsilon e^{S_n\phi(y) - nP(\phi)}.$$

Proof The shift $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ is an expansive homeomorphism. The existence of a unique equilibrium measure for the Holder potential $\phi \circ \pi$ with respect to the homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ follows from the standard theory of expansive homeomorphisms (for example [5]); let us denote it by $\hat{\mu}_\phi$. According to the discussion above there exists a unique probability measure μ_ϕ with $\mu_\phi := \pi_* \hat{\mu}_\phi$, and μ_ϕ is the unique equilibrium measure for ϕ on Λ. The uniqueness follows from the bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\hat{f})$ and from the fact that $\hat{\phi} := \phi \circ \pi : \hat{\Lambda} \to \mathbb{R}$ is Holder continuous (as $\pi : \hat{\Lambda} \to \Lambda$ is Lipschitz and ϕ is Holder). Now, there exists a $k = k(\varepsilon) \geq 1$ such that $\hat{f}^k(\pi^{-1} B_n(y, \varepsilon)) \subset B_{n-k}(\hat{f}^k \hat{y}, 2\varepsilon) \subset \hat{\Lambda}$, for any $y \in \Lambda$. On the other hand for any $\hat{y} \in \hat{\Lambda}$, we have $\pi(B_n(\hat{y}, \varepsilon)) \subset B_n(y, \varepsilon)$. The last two set inclusions and the $\hat{f}$-invariance of $\hat{\mu}_\phi$, together with the estimates for the $\hat{\mu}_\phi$-measure of the Bowen balls in $\hat{\Lambda}$ (from [5]) imply that there exist positive constants A_ε, B_ε (depending on $\varepsilon > 0$ and ϕ) such that the estimates from the statement hold. □

Next let us show that the notion of connected repellor is *stable under perturbations*; this property is important when dealing with systems having a small level of random noise, as it happens in most physical situations.

Proposition 3 *Let Λ be a connected repellor for a smooth map $f : M \to M$ so that f is hyperbolic on Λ, and let a perturbation g which is $\mathcal{C}^1$-close to f. Then g has a connected repellor Λ_g close to Λ such that g is hyperbolic on Λ_g. In addition the number of g-preimages belonging to Λ_g of any point of Λ_g, is the same as the number of f-preimages in Λ of a point from Λ.*

Proof Since Λ has a neighbourhood U so that $\bar{U} \subset f(U)$, it follows that for g close enough to f, we will obtain $\bar{U} \subset g(U)$. If g is $\mathcal{C}^1$-close to f, then we can take the set

$$\Lambda_g := \bigcap_{n \in \mathbb{Z}} g^n(U)$$

and it is quite standard that g is hyperbolic on Λ_g (for example [10, 18], etc.). One can form then the natural extension of the system (Λ_g, g). We know that there exists a conjugating homeomorphism $H : \hat{\Lambda} \to \hat{\Lambda}_g$ which commutes with $\hat{f}$ and $\hat{g}$. The natural extension $\hat{\Lambda}$ is connected if Λ is connected, from the fact that the topology on $\hat{\Lambda}$ is induced by the product topology from $\Lambda^{\mathbb{N}}$. Hence $\hat{\Lambda}_g$ is connected and thus Λ_g itself is connected too. Moreover we have that $\bar{U} \subset g(U)$ if g is close enough to f, thus Λ_g is a connected repellor for g.

Now for the second part of the proof, assume that $x \in \Lambda$ has d f-preimages in Λ. Then if $C_f \cap \Lambda = \emptyset$ and if g is $\mathcal{C}^1$-close enough to f, it follows that the local inverse iterates of g are close to the local inverse iterates of f near Λ. Thus any point $y \in \Lambda_g$ has exactly d g-preimages in U, denoted by $y_1, \dots, y_d$. Any of these g-preimages from U has also a g-preimage in U since $\bar{U} \subset g(U)$, and so on. This implies that $y_i \in \Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$, $i = 1, \dots, d$; hence y has exactly d g-preimages belonging to Λ_g. □

We will need in the sequel an estimate of the volume of a *tubular unstable neighbourhood* $f^n(B_n(y, \varepsilon))$, where $B_n(y, \varepsilon) := \{z \in M, d(f^i z, f^i y) < \varepsilon, i = 0, \dots, n-1\}$ is a Bowen ball. The set $f^n(B_n(y, \varepsilon))$ is a neighbourhood in M of the local unstable manifold $W^u_\varepsilon(\hat{f}^n y)$, for $\hat{f}^n y = (f^n y, f^{n-1} y, \dots, y, \dots)$. Such sets were used in the definition of the inverse pressure, a notion developed in order to obtain estimates for the stable dimension in the non-invertible case ([12]).

By the measure $m(\cdot)$ on M we understand the *Lebesgue measure* defined on the manifold M. And by $S_n\phi(y)$ we denote the consecutive sum $\phi(y) + \dots + \phi(f^{n-1} y)$ for $y \in \Lambda, \phi \in \mathcal{C}(\Lambda, \mathbb{R})$.

Lemma 1 *Let $f : M \to M$ be a smooth endomorphism and Λ be a basic set on which f is hyperbolic. Then for some fixed small $\varepsilon > 0$ there exist positive constants $A, B > 0$ such that for any $n \ge 1$ we have*:

$$Ae^{S_n \Phi^s(y)} \le m(f^n B_n(y, \varepsilon)) \le Be^{S_n \Phi^s(y)}.$$

Proof First of all let us notice that $S_n \Phi^s(y) = \log|\det(Df^n_s(y))|$, $y \in \Lambda, n \ge 1$. From [5] we know that the stable spaces depend Holder continuously on their base point. Thus Φ^s is a Holder function on Λ, as $\mathcal{C}_f \cap \Lambda = \emptyset$. Thus as in Proposition 1.6 from [12], we obtain a Bounded Distortion Lemma, saying that there exist positive constants $\tilde{A}, \tilde{B}$ such that $\tilde{A} \le \frac{e^{S_n \Phi^s(z)}}{e^{S_n \Phi^s(y)}} \le \tilde{B}, n \ge 1, z \in B_n(y, \varepsilon)$. Then using this Bounded Distortion Lemma, the conclusion follows similarly as in [2]. □

Theorem 1 *Consider Λ to be a connected hyperbolic repellor for the smooth endomorphism $f: M \to M$; let us assume that the constant number of f-preimages belonging to Λ, of any point from Λ, is equal to d. Then $P(\Phi^s - \log d) = 0$.*

Proof As we have seen in Proposition 1 if Λ is a connected repellor, then the number of preimages belonging to Λ of any point from Λ is constant and equal to some integer $d > 0$. In fact if the neighbourhood V of Λ is close enough to Λ, then we can assume that any point $y \in V$ has exactly d f-preimages belonging to U. We want to show that there exists a neighbourhood V of Λ such that any point from V has exactly d^n n-preimages belonging to U, for any $n \geq 1$. First let us assume that the metric around Λ is adapted to the hyperbolic structure on Λ, i.e. there is $\lambda \in (0, 1)$ so that if $z \in W^u_r(\hat{y})$ and $\hat{z} = (z, z_{-1}, \ldots)$ is the prehistory of z r-shadowing the prehistory $\hat{y}$, then

$$d(y, z) \geq d(y_{-1}, z_{-1}) \cdot \frac{1}{\lambda} \geq d(y_{-2}, z_{-2}) \cdot \frac{1}{\lambda^2} \geq \cdots. \tag{1}$$

Now consider a point $y \in V$ and some preimage $y_{-1} \in f^{-1}(y) \cap U$. If y is close enough to Λ, then $y_{-1} \in U$, and let us assume that we can continue this prehistory until we reach level m. In other words $(y, y_{-1}, \ldots, y_{-m})$ is a finite prehistory of y with $y_{-1}, \ldots, y_{-m} \in U$, but there exists a preimage $\tilde{y}_{-m-1}$ of y_{-m} which escapes U, so that y_{-m} has less than d preimages in U. From the definition of repellor we know that $\bar{U} \subset f(U)$, thus there exists some preimage $y_{-m-1} \in U \cap f^{-1}(y_{-m})$. Then this preimage y_{-m-1} will have a full prehistory in U. Since Λ is a basic set and y_{-m} has a full prehistory in U, it follows that there exists a prehistory $\hat{\xi} \in \hat{\Lambda}$ such that $y_{-m} \in W^u_r(\hat{\xi})$, if U is close enough to Λ ([5, 18]).

Consequently $y \in W^u_r(\hat{f}^m\hat{\xi})$; but from (1) we have $d(y_{-m}, \Lambda) \leq d(y_{-m}, \xi) \leq \lambda^m d(y, f^m\xi) \leq \lambda^m d(y, \Lambda)$. Recall however that a preimage of y_{-m} escapes U, thus $d(y_{-m}, \Lambda)$ must be larger than some positive fixed constant χ_0. Therefore if V is close enough to Λ (and hence m is large enough) we obtain a contradiction, since we know from above that $d(y_{-m}, \Lambda) \leq \lambda^m \cdot d(y, \Lambda)$.

Hence there must exist a neighbourhood V of Λ such that any point from V has exactly d^n n-preimages belonging to U, for any $n \geq 1$.

Let us take now an (n, ε)-separated set of maximal cardinality in Λ and denote it by $F_n(\varepsilon)$. Hence $B_n(y, \varepsilon/2) \cap B_n(z, \varepsilon/2) = \emptyset, \forall y, z \in F_n(\varepsilon)$. From the maximality condition it follows also that $\Lambda \subset \bigcap_{y \in F_n(\varepsilon)} B_n(y, 2\varepsilon)$. Now from the fact that $C_f \cap \Lambda = \emptyset$, it follows that there exists a positive constant ε_0 such that if $y, z \in f^{-1}x \cap U, y \neq z$, then $d(y, z) > \varepsilon_0$. This implies that if $y, z \in f^{-n}x \cap \Lambda, y \neq z$, then we cannot have $z \in B_n(y, 4\varepsilon)$ for small enough ε.

So for a point $y \in V$, we know that any two of its different n-preimages must belong to distinct balls of type $B_n(\zeta, 2\varepsilon), \zeta \in F_n(\varepsilon)$; and y must have d^n n-preimages in U. If y_{-n} is an n-preimage in U of y, then there exists $\hat{\xi} \in \hat{\Lambda}$ so that $y \in W^u_\varepsilon(\hat{\xi})$ and thus $y_{-n} \in B_n(\xi_{-n}, \varepsilon)$ for some $\xi_{-n} \in \Lambda$. But since $F_n(\varepsilon)$ is a maximal (n, ε)-separated set in Λ, it follows that $\xi_{-n} \in B_n(z, 2\varepsilon)$ for some $z \in F_n(\varepsilon)$. Hence $y_{-n} \in B_n(z, 3\varepsilon)$ and $y \in f^n(B_n(z, 3\varepsilon))$ for some $z \in F_n(\varepsilon)$. Thus we have the following geometric picture of the dynamics on the basin V of the repellor: through every point $y \in V$ there pass d^n tubular neighbourhoods of type $f^n B_n(z_i, 3\varepsilon), z_i \in F_n(\varepsilon), i = 1, \ldots, d^n$. Let us denote such an intersection by $V_n(z_1, \ldots, z_{d^n})$.

Therefore from Lemma 1 it follows that, if we add the volumes of all sets $f^n(B_n(z, 3\varepsilon))$, $z \in F_n(\varepsilon)$, we obtain that each piece $V_n(z_1, \ldots, z_{d^n})$ is repeated at least d^n times, hence

$$d^n m(V) \leq \sum_{z \in F_n(\varepsilon)} e^{S_n \Phi^s(z)}.$$

Thus since this happens for any maximal (n, ε)-separated set $F_n(\varepsilon)$,

$$m(V) \leq P_n(\Phi^s - \log d, \varepsilon), \tag{2}$$

where $P_n(\psi, \varepsilon)$ denotes in general the quantity $\inf\{\sum_{z\in F} e^{S_n\psi(z)}, F(n, \varepsilon) - \text{separated in } \Lambda\}$, for ψ a continuous real function on Λ.

Since V is a neighbourhood of Λ and thus $m(V) > 0$, we obtain that

$$P(\Phi^s - \log d) \geq 0.$$

We prove now the opposite inequality. Indeed let us take some maximal (n, ε)-separated set $F_n(\varepsilon)$ in Λ (with respect to f). Let a point $y \in V$, where the neighbourhood V of Λ was constructed earlier in the proof. Then similar to the above proof of the first inequality, each n-preimage y^i_{-n} of y must belong to some Bowen ball $B_n(z^i, 3\varepsilon), z^i \in F_n(\varepsilon), i = 1, \dots, d^n$; hence y belongs to the (open) intersection of d^n tubular unstable neighbourhoods centered at points $f^n(z_i), z_1, \dots, z_{d^n} \in F_n(\varepsilon)$, i.e. $y \in \bigcap_{1\leq i\leq d^n} f^n(B_n(z_i, 3\varepsilon))$. If y would belong also to some additional tubular unstable neighbourhood $f^n(B_n(\omega, 3\varepsilon))$ for some $\omega \in F_n(\varepsilon)$, besides the d^n neighbourhoods $f^n(B_n(z_i, 3\varepsilon)), i = 1, \dots, d^n$, then it would follow that y has an additional n-preimage $y^{d^n+1}_{-n} \in B_n(\omega, 3\varepsilon)$. Thus since $B_n(\omega, 3\varepsilon) \subset U$ for small $\varepsilon > 0$ and for $\omega \in \Lambda$, we would get a contradiction since y has at most d^n n-preimages in U; here we used again that Λ does not intersect the critical set of f. So any $y \in V$ belongs to only d^n tubular unstable neighbourhoods of type $f^n(B_n(z^i, 3\varepsilon)), i = 1, \dots, d^n$.

Now, as we see from Lemma 1, the Lebesgue measure of a tubular unstable neighbourhood $f^n(B_n(z, 3\varepsilon)), z \in \Lambda$ is comparable to $e^{S_n\Phi^s(z)}$ (where by *comparable* we mean that the ratio of the two quantities is bounded below and above by positive constants which are independent of z, n). Hence we showed that by taking $\sum_{z\in F_n(\varepsilon)} e^{S_n\Phi^s(z)}$ we cover in fact a combined volume which is less than $Cd^n \cdot m(U)$ (for some positive constant C independent of n). From this observation it follows that

$$P(\Phi^s - \log d) \leq \inf_n \frac{1}{n} m(U) = 0.$$

Combining the two inequalities proved above, we obtain that $P(\Phi^s - \log d) = 0$. □

We are now ready to prove the main result of the paper, namely the existence of a physical measure for the local inverse iterates in the neighbourhood V of the hyperbolic repellor Λ. We recall that the endomorphism f is *not* assumed to be expanding on Λ, instead it has both stable and unstable directions on Λ. As seen earlier, we can restrict without loss of generality to connected repellors. Recall also that we assumed that the critical set of f does not intersect Λ.

Theorem 2 *Let Λ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$. There exists a neighbourhood V of Λ, $V \subset U$ such that if we denote by*

$$\mu^z_n := \frac{1}{n} \sum_{y\in f^{-n}z\cap U} \frac{1}{d(f(y)) \cdot \dots \cdot d(f^n(y))} \sum_{i=1}^n \delta_{f^i y}, \quad z \in V,$$

where $d(y)$ is the number of f-preimages belonging to U of a point $y \in V$, then for any continuous function $g \in \mathcal{C}(U, \mathbb{R})$ we have

$$\int_V |\mu^z_n(g) - \mu_s(g)| dm(z) \underset{n\to\infty}{\to} 0,$$

where μ_s is the equilibrium measure of the stable potential $\Phi^s(x) := \log|\det(Df_s(x))|$, $x \in \Lambda$.

Proof We assume that U is the neighbourhood of Λ from Definition 1, i.e. such that $\bar{U} \subset f(U)$. As we proved in Proposition 1, if Λ is a connected hyperbolic repellor, then any point from Λ has exactly d f-preimages belonging to Λ for some positive integer d. Moreover as was shown in the beginning of the proof of Theorem 1, there exists a neighbourhood V of Λ such that any point from V has d^n n-preimages in U, for $n \geq 1$.

If Λ is a hyperbolic repellor we have that all local stable manifolds must be contained in Λ. Indeed, otherwise there may exist small local stable manifolds which are not entirely contained in Λ. Let $W^s_r(x)$, $x \in \Lambda$ one such stable manifold, with a point $y \in W^s_r(x) \setminus \Lambda$; in this case since $y \in U$ (for small r) and since $\bar{U} \subset f(U)$, it follows that y has a full prehistory $\hat{y}$ in U, and from the fact that Λ is a basic set, we obtain that $y \in W^u_r(\hat{\xi})$ for some $\hat{\xi} \in \hat{\Lambda}$. But then $y = W^s_r(x) \cap W^u_r(\hat{\xi})$, hence $y \in \Lambda$ from the local product structure of Λ (since Λ is a basic set, see for example [5]); this gives a contradiction to our assumption. Hence there exists a small $r > 0$ such that all stable manifolds of size r are contained in Λ.

We shall denote by $\mathcal{C}(U)$ the space of real continuous functions on U. Let us fix now a Holder continuous function $g \in \mathcal{C}(U)$. We will apply the L^1 Birkhoff Ergodic Theorem ([8]) on $\hat{\Lambda}$ for the homeomorphism $\hat{f}^{-1}$, in order to obtain an estimate for the measure of the set of prehistories which are badly behaved. Similarly as in [5] or [12] we know that the stable distribution is Holder continuous, hence the stable potential on $\hat{\Lambda}$ is Holder too. This means that there exists a unique equilibrium measure for this potential on $\hat{\Lambda}$; so from the bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\hat{f})$ it follows that there exists a unique equilibrium measure for Φ^s on Λ denoted by μ_s. This measure is ergodic and we can apply the L^1 Birkhoff Ergodic Theorem to the function $g \circ \pi$ on $\hat{\Lambda}$:

$$\left\| \frac{1}{n}(g(x) + g \circ \pi(\hat{f}^{-1}(\hat{x})) + \cdots + g \circ \pi(\hat{f}^{-n+1}(\hat{x})) - \int_\Lambda g \circ \pi \, d\hat{\mu}_s \right\|_{L^1(\hat{\Lambda}, \hat{\mu}_s)} \underset{n\to\infty}{\to} 0. \quad (3)$$

We make now the general observation that if $f : \Lambda \to \Lambda$ is a continuous map on a compact metric space Λ, μ is an f-invariant Borelian probability measure on Λ and $\hat{\mu}$ is the unique $\hat{f}$-invariant probability measure on $\hat{\Lambda}$ with $\pi_*(\hat{\mu}) = \mu$, then for an arbitrary closed set $\hat{F} \subset \hat{\Lambda}$, we have that

$$\hat{\mu}(\hat{F}) = \lim_n \mu(\{x_{-n}, \exists \hat{x} = (x, \dots, x_{-n}, \dots) \in \hat{F}\}). \quad (4)$$

Let us prove (4): first denote $\hat{F}_n := \hat{f}^{-n}\hat{F}, n \geq 1$; next notice that $\hat{\mu}(\hat{F}_n) = \hat{\mu}(\hat{F})$ since $\hat{\mu}$ is $\hat{f}$-invariant. Let also $\hat{G}_n := \pi^{-1}(\pi(\hat{F}_n)), n \geq 1$. We have $\hat{F} \subset \hat{f}^n(\hat{G}_n), n \geq 0$. Let now a prehistory $\hat{z} \in \bigcap_{n\geq 0} \hat{f}^n \hat{G}_n$; then if $\hat{z} = (z, z_{-1}, \dots, z_{-n}, \dots)$, we obtain that $z_{-n} \in \pi\hat{F}_n, \forall n \geq 0$, hence $\hat{z} \in \hat{F}$ since $\hat{F}$ is assumed closed. Thus we obtain $\hat{F} = \bigcap_{n\geq 0} \hat{f}^n(\hat{G}_n)$. Now the above intersection is decreasing, since $\hat{f}^{n+1}\hat{G}_{n+1} \subset \hat{f}^n\hat{G}_n, n \geq 0$. Since the above intersection is decreasing, we get that $\hat{\mu}(\hat{F}) = \lim_n \hat{\mu}(\hat{f}^n\hat{G}_n) = \lim_n \hat{\mu}(\hat{G}_n) = \lim_n \hat{\mu}(\pi^{-1}(\pi(\hat{F}_n))) = \lim_n \mu(\pi(\hat{F}_n)) = \lim_n \mu(\pi \circ \hat{f}^{-n}\hat{F})$, since $\hat{\mu}$ is $\hat{f}$-invariant. Therefore we obtain (4).

For a positive integer n, a continuous real function g defined on the neighbourhood U of Λ, and a point y so that $y, f(y), \dots, f^{n-1}(y)$ are all in U, let us denote by

$$\Sigma_n(g, y) := \frac{g(y) + \cdots + g(f^{n-1}y)}{n} - \int g \, d\mu_s, \quad n \geq 1, \; y \in \Lambda.$$

Now from the convergence in $L^1(\hat{\Lambda}, \hat{\mu}_s)$ norm established in (3), it follows the convergence in $\hat{\mu}_s$-measure; i.e. if we consider for a small $\eta > 0$ and an integer $n > 1$ the closed set:

$$\hat{F}_n(\eta) = \{\hat{x} = (x, x_{-1}, x_{-2}, \dots) \in \hat{\Lambda}, |\Sigma_n(g, x_{-n})| \geq \eta\},$$

then we have the convergence

$$\hat{\mu}_s(\hat{F}_n(\eta)) \underset{n\to\infty}{\to} 0, \quad \forall \eta > 0. \tag{5}$$

Thus from (5), (4) and the f-invariance of μ_s, we obtain that for any small $\eta > 0, \chi > 0$ there exists an integer $N(\eta, \chi) \geq 1$ so that:

$$\mu_s(x_{-n'} \in \Lambda \cap f^{-n'+n}(x_{-n}), |\Sigma_n(g, x_{-n})| \geq \eta) = \mu_s(x_{-n} \in \Lambda, |\Sigma_n(g, x_{-n})| \geq \eta) < \chi, \tag{6}$$

for $n' > n > N(\eta, \chi)$.

Let us consider now some small $\varepsilon > 0$. Recall that for $n \geq 1$ and $y \in \Lambda$, the Bowen ball $B_n(y, \varepsilon) := \{z \in M, d(f^i y, f^i z) < \varepsilon, i = 0, \dots, n-1\}$. We shall prove that if $y \in \Lambda$ and $z \in B_n(y, \varepsilon)$ for n large enough, then the behaviour of $\Sigma_n(g, z)$ is similar to that of $\Sigma_n(g, y)$. More precisely, assume that $\eta > 0$ and that $y \in \Lambda$ satisfies $|\Sigma_n(g, y)| \geq \eta$. Then we will show that there exists $N(\eta) \geq 1$ so that

$$|\Sigma_n(g, z)| \geq \frac{\eta}{2}, \quad \forall z \in B_n(y, \varepsilon), n > N(\eta). \tag{7}$$

Since g was assumed Holder, let us assume that it has a Holder exponent equal to α, i.e.

$$|g(x) - g(y)| \leq C \cdot d(x, y)^\alpha, \quad \forall x, y \in U,$$

where $d(x, y)$ is the Riemannian distance (from M) between x and y and $C > 0$ is a constant. The idea now is that, if $z \in B_n(y, \varepsilon)$, then for some time the iterates of z follow the iterates of y close to stable manifolds, and afterwards they follow the iterates of y closer and closer to unstable manifolds. We have in both cases an exponential growth of distances between iterates, and thus we can use the Holder continuity of g on U.

If $z \in B_n(y, \varepsilon), y \in \Lambda$ then we either have $z \in W^s_\varepsilon(y) \subset \Lambda$ or there exists a positive distance between z and the local stable manifold $W^s_\varepsilon(y)$. In the first case there exists some $\lambda_s \in (0, 1)$ such that $d(f^i z, f^i y) < \lambda_s^i \varepsilon, i = 0, \dots, n-1$. This implies that, in the case when $z \in W^s_\varepsilon(y)$, for some $N_0 \geq 1$ we have:

$$|g(f^{N_0} y) + \dots + g(f^{n-1} y) - g(f^{N_0} z) - \dots - g(f^{n-1} z)| \leq \lambda_s^{\alpha N_0} \cdot C_0, \tag{8}$$

for some constant $C_0 > 0$ independent of n. If $z \in B_n(y, \varepsilon)$ but z is not necessarily on $W^s_\varepsilon(y)$, then the iterates of z will approach exponentially some local unstable manifolds at the corresponding iterates of y and their "projections" on these unstable manifolds increases exponentially, up to a maximum value less than ε (reached at level n). More precisely there exists some $N_0, N_1 \geq 1$ and some $\lambda \in (\lambda_s, 1)$ such that $d(f^i z, f^i y) \leq \lambda^i, i = N_0, \dots, N_1 - 1$; notice that N_0, N_1, λ are independent of y, z, n. Now if the iterate $f^{N_1} z$ becomes much closer to $W^u_\varepsilon(f^{N_1} y)$ than to $W^s_\varepsilon(f^{N_1} y)$, it follows that all the higher order iterates will approach asymptotically the local unstable manifolds and $d(f^j y, f^j z)$ increases exponentially. We assume that N_1 has been taken such that for some $\lambda_u \in (\frac{1}{\inf_\Lambda |Df_u|}, 1)$, we have $d(f^j z, f^j y) \leq \lambda_u \cdot d(f^{j+1} z, f^{j+1} y), j = N_1, \dots, n-2$. So the maximum such distance is $d(f^{n-1} y, f^{n-1} z)$ and we know that $d(f^{n-1} y, f^{n-1} z) < \varepsilon$ since $z \in B_n(y, \varepsilon)$. Hence

$$d(f^j z, f^j y) \leq \varepsilon \lambda_u^{n-j-1}, \quad j = N_1, \dots, n-1.$$

Let us take now some $N_2 \geq 1$ such that $n - N_2 > N_1$; N_2 will be determined later. Thus from the Holder continuity of g on U we obtain (for some positive constant C) that:

$$\begin{aligned}
&|g(f^{N_0}z) + \cdots + g(f^{N_1-1}z) + g(f^{N_1}z) + \cdots + g(f^{n-N_2}z) + \cdots + g(f^{n-1}z) \\
&\quad - g(f^{N_0}y) - \cdots - g(f^{N_1-1}y) - g(f^{N_1}y) - \cdots - g(f^{n-N_2}y) - \cdots - g(f^{n-1}y)| \\
&\leq C(\lambda^{\alpha N_0} + \lambda_u^{\alpha N_2}) + 2N_2\|g\|. \qquad (9)
\end{aligned}$$

Thus from (8) and (9) we obtain that, if $z \in B_n(y, \varepsilon)$ then:

$$|\Sigma_n(g, y) - \Sigma_n(g, z)| \leq \frac{1}{n}\left[2N_0\|g\| + C(\lambda^{\alpha N_0} + \lambda_u^{\alpha N_2}) + 2N_2\|g\|\right]. \qquad (10)$$

From above, N_0, N_2 do not depend on n, y, z. Therefore we can choose some large $N(\eta)$ so that

$$\frac{1}{n}(2N_0\|g\| + C(\lambda^{\alpha N_0} + \lambda_u^{\alpha N_2}) + 2N_2\|g\|) < \eta/2, \quad \text{for } n > N(\eta).$$

This means that the relation from (7) holds. Let us denote now by:

$$I_n(g, x) := \frac{1}{d^n} \sum_{y \in f^{-n}(x) \cap U} |\Sigma_n(g, y)|, \qquad (11)$$

for a continuous real function $g : U \to \mathbb{R}$, and $x \in V$. Recall that V is the neighbourhood of Λ, $\Lambda \subset V \subset U$, constructed in the proof of Theorem 2 so that every point $x \in V$ has d^n n-preimages in U for $n \geq 1$. For a fixed Holder continuous function g and a small $\eta > 0$, we will work with $n > N(\eta)$, where $N(\eta)$ was found above. From (10) and the discussion afterwards, we know that $|\Sigma_n(g, z) - \Sigma_n(g, y)| \leq \eta/2$ if $z \in B_n(y, \varepsilon)$ and $y \in \Lambda$.

Let us consider now an (n, ε)-separated set with maximal cardinality in Λ, denoted by $F_n(\varepsilon)$. As in the proof of Theorem 1 it follows that any point $y \in V$ belongs to d^n tubular neighbourhoods, i.e. $f^n(B_n(y_i, 3\varepsilon))$, $y_i \in F_n(\varepsilon)$ for $1 \leq i \leq d^n$. Let us denote as before $V_n(y_1, \dots, y_{d^n}) := \bigcap_{1 \leq i \leq d^n} f^n B_n(y_i, 3\varepsilon)$. Thus in the integral $\int_V I_n(g, x)dm(x)$, we can decompose V into the smaller pieces $V_n(y_1, \dots, y_{d^n})$, for different choices of $y_1, \dots, y_{d^n} \in F_n(\varepsilon)$.

We can use now relation (10) in order to replace in $\int_V I_n(g, x)dm(x)$, the term $|\Sigma_n(g, y)|$ with $|\Sigma_n(g, \zeta)|$, where $x \in V$ is arbitrary, $y \in f^{-n}x \cap U$ and $y \in B_n(\zeta, 3\varepsilon)$ for some $\zeta \in F_n(\varepsilon)$. Indeed let us fix some arbitrary small $\eta > 0$. Then we prove similarly as in (10) that if $n > N(\eta)$, then $|\Sigma_n(g, y)| \leq |\Sigma_n(g, \zeta)| + \eta/2$, if $y \in B_n(\zeta, 3\varepsilon)$ and $\zeta \in F_n(\varepsilon)$ ($N(\eta)$ can be assumed to be the same as in (10) without loss of generality).

So up to a small error of $\eta/2$ we can replace each of the terms $|\Sigma_n(g, y)|$ with the corresponding $|\Sigma_n(g, \zeta)|$. This implies that in the integral $\int_V I_n(g, x)dm(x)$, on each piece of type $V_n(y_1, \dots, y_{d^n})$ in $f^n(B_n(y_j, 3\varepsilon))$ for $y_j \in F_n(\varepsilon)$, we integrate in fact $|\Sigma_n(g, y_j)|$, modulo an error of $\eta/2$. Then we will obtain that

$$\begin{aligned}
\int_V I_n(g, x)dm(x) &\leq \frac{1}{d^n} \sum_{z_1, \dots, z_{d^n} \in F_n(\varepsilon)} \int_{V_n(z_1, \dots, z_{d^n})} \sum_{i=1}^{n} |\Sigma_n(g, z_i)| dm + \frac{\eta}{2} \cdot m(V) \\
&\leq \frac{1}{d^n} \sum_{z \in F_n(\varepsilon)} |\Sigma_n(g, z)| \cdot \sum_{z \in \{z_1, \dots, z_{d^n}\}} m(V_n(z_1, \dots, z_{d^n})) + m(V)\eta/2
\end{aligned}$$

$$\leq \frac{1}{d^n} \sum_{z \in F_n(\varepsilon)} |\Sigma_n(g,z)| \cdot m(f^n B_n(z,3\varepsilon)) + m(V)\eta/2.$$

So what we did is, we replaced $|\Sigma_n(g,y)|$ with $|\Sigma_n(g,z)|$ for all $y \in f^{-n}x \cap U$, where $y \in B_n(z,3\varepsilon), z \in F_n(\varepsilon)$, then we integrated the respective sums of $|\Sigma_n(g,z)|, z \in F_n(\varepsilon)$ on small pieces of tubular overlap $V_n(z_1,\dots,z_{d^n})$; lastly, we kept $|\Sigma_n(g,z)|$ fixed for an arbitrary $z \in F_n(\varepsilon)$ and added the measures of all intersections of $f^n B_n(z,3\varepsilon)$ with other tubular sets of type $f^n B_n(w,3\varepsilon), w \in F_n(\varepsilon)$. Thus by adding the measures of these overlaps, we recover $m(f^n B_n(z,3\varepsilon))$. In conclusion we obtain:

$$\int_V I_n(g,x)dm(x) \leq C \cdot \sum_{y \in F_n(\varepsilon)} |\Sigma_n(g,y)| \cdot \frac{m(f^n(B_n(y,3\varepsilon))}{d^n} + \frac{\eta}{2} \cdot m(V). \tag{12}$$

We recall now from Lemma 1, that $m(f^n(B_n(y,3\varepsilon)))$ is comparable to $e^{S_n\Phi^s(y)}$, independently of $n, y \in \Lambda$. And from Theorem 1 we know that $P(\Phi^s) = \log d$. Hence from Proposition 2 we have that, if μ_s denotes the unique equilibrium measure of Φ^s, then $\mu_s(B_n(y,\varepsilon/2))$ is comparable to $\frac{e^{S_n\Phi^s(y)}}{d^n}$, independently of n, y. Therefore combining with (12) we obtain that there exists a constant $C_1 > 0$ s.t:

$$\int_V I_n(g,x)dm(x) \leq C_1 \left(\sum_{y \in F_n(\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) + \eta \right), \tag{13}$$

for $n > N(\eta)$. We will split now the points of $F_n(\varepsilon)$ in two disjoint subsets denoted by $G_1(n,\varepsilon)$ and $G_2(n,\varepsilon)$, defined as follows:

$$G_1(n,\varepsilon) := \{y \in F_n(\varepsilon), |\Sigma_n(g,y)| < \eta\} \quad \text{and} \quad G_2(n,\varepsilon) := \{z \in F_n(\varepsilon), |\Sigma_n(g,z)| \geq \eta\}.$$

Recall that the Bowen balls $B_n(y,\varepsilon/2), y \in F_n(\varepsilon)$ are mutually disjointed since $F_n(\varepsilon)$ is (n,ε)-separated. Also if $y \in G_2(n,\varepsilon)$ and $z \in B_n(y,\varepsilon/2)$, we have $|\Sigma_n(g,z)| \geq \eta/2$ (from (7)); hence $B_n(y,\varepsilon/2) \cap \Lambda \subset \{z \in \Lambda, |\Sigma_n(g,z)| \geq \eta/2\}$. Consequently for a constant $C_\varepsilon > 0$,

$$\begin{aligned}
&\sum_{y \in F_n(\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) \\
&\quad = \sum_{y \in G_1(n,\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) + \sum_{y \in G_2(n,\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) \\
&\quad \leq \eta \sum_{y \in G_1(n,\varepsilon)} \mu_s(B_n(y,\varepsilon/2)) + 2\|g\| \mu_s\left(z \in \Lambda, |\Sigma_n(g,z)| \geq \frac{\eta}{2}\right) \cdot C_\varepsilon.
\end{aligned}$$

But since the balls $B_n(y,\varepsilon/2), y \in F_n(\varepsilon)$ are mutually disjoint, we have $\sum_{y \in G_1(n,\varepsilon)} \mu_s(B_n(y,\varepsilon/2)) \leq 1$. Also $\mu_s(z \in \Lambda, |\Sigma_n(g,z)| \geq \eta/2) < \chi$ for $n > N(\eta/2,\chi)$, as follows from (6). Thus by using (13) we obtain for $n > \sup\{N(\eta), N(\eta,\chi)\}$

$$\int_V I_n(g,x)dm(x) \leq C_1(\eta + \eta + C_\varepsilon \cdot 2\|g\|\chi) = 2C_1(\eta + \chi \cdot C_\varepsilon \|g\|).$$

Since $\eta, \chi > 0$ were taken arbitrarily, and by recalling the formula for $I_n(g, x)$ from (11) and the definition of μ_n^z, we obtain that:

$$\int_V |\mu_n^z(g) - \mu_s(g)| dm(z) \underset{n\to\infty}{\to} 0.$$

Since Holder continuous functions g are dense in the uniform norm on $\mathcal{C}(U)$, we obtain the conclusion of the Theorem for all $g \in \mathcal{C}(U)$. □

Corollary 1 *In the same setting as in Theorem 2, it follows that there exists a Borelian set $A \subset V$ with $m(V \setminus A) = 0$ and a subsequence $(n_k)_k$ such that $\mu_{n_k}^z \to_{k\to\infty} \mu_s$ (as measures on U), for any point $z \in A$.*

Proof Let us fix $g \in \mathcal{C}(U)$. From the convergence in Lebesgue measure of the sequence of functions $z \to \mu_n^z(g), n \geq 1, z \in V$ obtained from Theorem 2, it follows that there exists a Borelian set $A(g)$ with $m(V \setminus A(g)) = 0$ and a subsequence $(n_p)_p$ so that $\mu_{n_p}^z(g) \to_p \mu_s(g), z \in A(g)$. Let us consider now a sequence of functions $(g_m)_{m\geq 1}$ dense in $\mathcal{C}(U)$. By applying a diagonal sequence procedure we obtain a subsequence $(n_k)_k$ so that $\mu_{n_k}^z(g_m) \to_k \mu_s(g_m), \forall z \in \bigcap_m A(g_m), \forall m \geq 1$. We have also $m(V \setminus \bigcap_m A(g_m)) = 0$, since $m(V \setminus A(g_m)) = 0, m \geq 1$. However any real continuous function $g \in \mathcal{C}(U)$ can be approximated in the uniform norm by functions g_m, hence it follows that $\mu_{n_k}^z(g) \to_k \mu_s(g), \forall z \in A := \bigcap_m A(g_m)$. But we showed above that $m(V \setminus A) = 0$. So we obtain that $\mu_{n_k}^z \to_k \mu_s$ for all points $z \in A$, where A has full Lebesgue measure in V. □

3 Ergodic Properties of the Inverse SRB Measure. Examples

In this section we will pursue further ergodic properties of the inverse physical measure constructed in Theorem 2 and give also examples. Let us first remind the notion of the Jacobian of an endomorphism, relative to an invariant probability measure, from Parry's book ([14]). Let $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ a measure preserving endomorphism on a Lebesgue probability space. Assume that the fibers of f are countable, i.e. $f^{-1}x$ is countable for μ-almost all $x \in X$. It can be proved ([14]) that in this case f is positively non-singular, i.e. $\mu(A) = 0$ implies $\mu(f(A)) = 0$ for an arbitrary measurable set $A \subset X$. Also there exists a measurable partition $\alpha = (A_0, A_1, \dots)$ of X such that $f|_{A_i}$ is injective. Then using the absolute continuity of $\mu \circ f$ with respect to μ, we define the *Jacobian* $J_{f,\mu}$ on each set A_i, to be equal to the Radon-Nikodym derivative $\frac{d\mu\circ f}{d\mu}$. So:

$$J_{f,\mu}(x) = \frac{d\mu \circ f}{d\mu}(x), \quad x \in A_i, i \geq 0.$$

This is a well defined measurable function, which is larger or equal than 1 everywhere (due to the f-invariance of μ). Also it is easy to see that $J_{f,\mu}(\cdot)$ is independent of the partition α and that it satisfies a Chain Rule, namely $J_{f\circ g,\mu} = J_{f,\mu} \cdot J_{g,\mu}$ if $f, g : X \to X$ and both preserve μ. From Lemma 10.5 of [14] we also know that

$$\log J_{f,\mu}(x) = I(\epsilon/f^{-1}\epsilon)(x),$$

for μ-almost every $x \in X$, where ϵ is the partition of X into single points, and $I(\epsilon/f^{-1}\epsilon)(\cdot)$ is the conditional information function of ϵ given the partition $f^{-1}\epsilon$. Also from the definition of the Jacobian we see ([7]) that:

$$\mu(fA) = \int_A J_{f,\mu}(x) d\mu(x), \tag{14}$$

for all *special sets* A, i.e. measurable sets such that $f|_A : A \to f(A)$ is injective. Recall that by Definition 1, f does not have any critical points in Λ. Before proving the main result of this Section, we remind the notion of measurable partitions subordinated to local stable manifolds; for background on measurable partitions, Lebesgue spaces and conditional measures, one can use [15].

Let $f : M \to M$ be a smooth endomorphism defined on a Riemannian manifold M which is endowed with its Borelian σ-algebra $\mathcal{B}$. Let also a probability Borelian measure μ on M which is f-invariant. If ξ is a measurable partition of M, then we denote by $\xi(x)$ the unique subset of ξ containing $x \in X$; also by $(M/\xi, \mu_\xi)$ we denote the factor space relative to ξ. To any measurable partition ξ on $(M, \mathcal{B}, \mu)$ one can attach an essentially unique collection of *conditional measures* $\{\mu_C\}_{C \in \xi}$ satisfying two conditions (see [15]):

(i) (C, μ_C) is a Lebesgue space
(ii) for any measurable set $B \subset M$, the set $B \cap C$ is measurable in C for almost all points $C \in M/\xi$ of the factor space, and the function $C \to \mu_C(B \cap C)$ is measurable on M/ξ and $\mu(B) = \int_{M/\xi} \mu_C(B \cap C) d\mu_\xi$.

Similar to the case of partitions subordinated to unstable manifolds ([21]) we can say (as in [7]), that a measurable partition ξ of $(M, \mathcal{B}, \mu)$ is *subordinate to local stable manifolds* if for μ-almost all $x \in M$ one has $\xi(x) \subset W^s_r(x)$ and if $\xi(x)$ contains an open neighbourhood of x inside $W^s_r(x)$ (where $r > 0$ is sufficiently small). We can define now the absolute continuity of conditional measures on stable manifolds as in [7]:

Definition 2 In the above setting, we say that μ has *absolutely continuous conditional measures on local stable manifolds* if for every measurable partition ξ subordinated to local stable manifolds, we have for μ almost all $x \in M$ that $\mu^\xi_x \ll m^s_x$, where μ^ξ_x is the conditional measure of μ on $\xi(x)$ and m^s_x denotes the induced Lebesgue measure on $W^s_r(x)$.

By the result of Liu ([7]), we know that there exists at least one measurable partition subordinated to local stable manifolds.

Now, by Oseledec Theorem ([8]) we have that for any f-invariant Borel probability measure μ on M, and for μ-almost every point $x \in M$ there exists a finite collection of numbers, called *Lyapunov exponents* of f at x with respect to μ, $-\infty \le \lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{q(x)}(x) < \infty$, and a unique collection of tangent subspaces of T_xM, $V_1(x) \subset \cdots \subset V_{q(x)}(x) = T_xM$ so that

$$\lim_n \frac{1}{n} \log |Df^n_x(v)| = \lambda_i(x), \quad \forall v \in V_i(x) \setminus V_{i-1}(x), 1 \le i \le q(x), |v| = 1.$$

We also denote by $m_i(x) := \dim V_i(x) - \dim V_{i-1}(x)$ the *multiplicity* of $\lambda_i(x)$. As we saw before, if Λ is a connected repellor for f then $f|_\Lambda$ is constant-to-1. We are now ready to prove the following:

Theorem 3 *Let Λ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$ on a Riemannian manifold M; assume that f is d-to-1 on Λ. Then there exists a unique f-invariant probability measure μ^- on Λ satisfying an inverse Pesin entropy formula*:

$$h_{\mu^-}(f) = \log d - \int_\Lambda \sum_{i,\lambda_i(x)<0} \lambda_i(x) m_i(x) d\mu^-(x).$$

In addition the measure μ^- has absolutely continuous conditional measures on local stable manifolds.

Proof Notice that from the above properties of Lyapunov exponents, the derivative $Df^n_{s,x}$ for large n, takes into consideration all the vectors $v \in V_i(x)$ for those i for which $\lambda_i(x) < 0$, i.e. for which we have contraction in the long run. Thus if μ is an f-invariant probability measure supported on Λ, we have by the Chain Rule and Birkhoff Theorem that

$$\begin{aligned}\int_\Lambda \Phi^s d\mu &= \int_\Lambda \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \Phi^s(f^i x) d\mu(x) \\ &= \int_\Lambda \lim_n \frac{1}{n} \log|\det(Df^n_{s,x})| d\mu(x) = \int_\Lambda \sum_{i,\lambda_i(x)<0} \lambda_i(x) m_i(x) d\mu(x). \quad (15)\end{aligned}$$

It follows that the inverse Pesin entropy formula from the statement of the Theorem is satisfied for $\mu = \mu_s$ since μ_s is the equilibrium measure of Φ^s and we showed in Theorem 1 that $P(\Phi^s - \log d) = 0$. If the inverse Pesin entropy formula would be satisfied for another invariant measure μ, then we would have $h_\mu(f) = \log d - \int_\Lambda \sum_{i,\lambda_i(x)<0} \lambda_i(x) m_i(x) d\mu(x)$, hence:

$$P(\Phi^s - \log d) \geq h_\mu - \log d + \int_\Lambda \Phi^s d\mu = 0.$$

However again from Theorem 1 we know that $P(\Phi^s - \log d) = 0$, thus μ is an equilibrium measure for Φ^s. But Φ^s is Holder continuous and thus it has a unique equilibrium measure. Therefore if $\mu^- := \mu_s$, we have

$$\mu = \mu_s = \mu^-.$$

We want now to show the absolute continuity of conditional measures of μ^- on local stable manifolds. For this we will use Corollary 1 and results from [7]. Indeed we know that Λ is a connected hyperbolic repelor and thus f is d-to-1 for some integer $d \geq 1$ in a neighbourhood V of Λ. We constructed the measures $\mu^z_n, z \in V, n \geq 1, \mu^z_n := \frac{1}{d^n} \sum_{y \in f^{-n}z} \frac{1}{n} \sum_{i=1}^n \delta_{f^i y}$; and we showed in Corollary 1 that there exists a subset $A \subset V$, having full Lebesgue measure and a subsequence $(\mu^z_{n_k})_k$ converging weakly towards $\mu^- := \mu_s$ for every $z \in A$. Now in (14) we can take only special sets whose boundaries have μ^--measure equal to zero. For such a set B we have that $\mu_{n_k}(B) \to_{k\to\infty} \mu^-(B)$. But then from the definition of μ^z_n it follows that $\mu^-(f(B)) = d\mu^-(B)$ for any such special set with boundary of measure zero. As these sets form a sufficient collection ([5]), we obtain that the Jacobian J_{f,μ^-} is constant μ^--almost everywhere and equal to d. Hence from Lemma 10.5 of [14], if ϵ denotes the partition of M into single points, we deduce that the conditional information function $I(\epsilon/f^{-1}\epsilon)(x) = \log J_{f,\mu^-}(x) = \log d$ for μ^--almost all $x \in \Lambda$; thus

$$H_{\mu^-}(\epsilon/f^{-1}\epsilon) = \int I(\epsilon/f^{-1}\epsilon)(x) d\mu^-(x) = \log d.$$

Then since $h_{\mu^-} = \log d - \int_\Lambda \sum_{\lambda_i(x)<0} \lambda_i(x) d\mu^-(x)$, it follows that

$$h_{\mu^-} = H_{\mu^-}(\epsilon/f^{-1}\epsilon) - \int_\Lambda \sum_{i,\lambda_i(x)<0} \lambda_i(x) m_i(x) d\mu^-(x).$$

Hence from [7] we obtain that μ^- has absolutely continuous conditional measures on local stable manifolds. □

We study now whether the measure-preserving endomorphism $(\Lambda, f|_\Lambda, \mathcal{B}(\Lambda), \mu^-)$ is isomorphic to a 1-sided Bernoulli transformation. For definitions and properties of 1-sided Bernoulli maps, see for example [2, 3, 15], etc. The question of whether a dynamical system with an invariant probability measure is Bernoulli (1-sided or 2-sided) is very important and was solved in a number of cases ([3, 8]). It gives a coding of the system from the point of view of the measure. For example for diffeomorphisms, it was shown that a hyperbolic attractor with its unique SRB measure is 2-sided Bernoulli ([2, 8, 20]). Also, for expanding maps on a space X, there exist Markov partitions ([19]); this implies that X, together with the equilibrium measure of any Holder potential, is measure-isomorphic to a 1-sided Markov chain ([17, 19]). However in our non-invertible non-expanding case we will see that this is not true anymore and that $(\Lambda, f|_\Lambda, \mathcal{B}(\Lambda), \mu^-)$ *is not 1-sided Bernoulli*. Here $\mathcal{B}(\Lambda)$ denotes the σ-algebra of Borelian sets in Λ.

Theorem 4 *Let f as above and Λ a connected repellor as in Theorem 3 so that f is not invertible on Λ. Then $(\Lambda, f|_\Lambda, \mathcal{B}(\Lambda), \mu^-)$ cannot be one-sided Bernoulli.*

Proof Let $(\Sigma_m^+, \sigma_m, \mathcal{B}_m, \mu_p)$ a one-sided Bernoulli shift on m symbols ([8]), where $\mathcal{B}_m$ denotes the σ-algebra of sets generated by cylinders in Σ_m^+, σ_m is the shift map, and μ_p is the σ_m-invariant measure associated to a probability vector $p = (p_1, \ldots, p_m)$.

We know from Proposition 1 that if Λ is connected, then the number of f-preimages belonging to Λ is constant, and denote it by d; we assumed that $d > 1$. If $(\Lambda, f|_\Lambda, \mathcal{B}(\Lambda), \mu^-)$ would be isomorphic to a one-sided Bernoulli system $(\Sigma_m^+, \sigma_m, \mathcal{B}_m, \mu_p)$, then $d = m$ since the number of preimages is constant everywhere, for both systems. But then from the Variational Principle for entropy, we would obtain:

$$h_{\mu^-} = h_{\mu_p} \le h_{top}(\sigma_m) = \log m = \log d. \tag{16}$$

On the other hand since μ^- satisfies the Pesin formula on Λ, we get that $h_{\mu^-} = \log d - \int \Phi^s d\mu^-$. But $\Phi^s < 0$ and $\mathcal{C}_f \cap \Lambda = \emptyset$, hence $h_{\mu^-} > \log d$. This gives a contradiction to (16). □

We prove now that, in spite of not being 1-sided Bernoulli, the inverse SRB measure μ^- has strong mixing properties on the repellor Λ.

Given a transformation $f : M \to M$ we say that an f-invariant probability μ has *Exponential Decay of Correlations* on Holder potentials ([2]) if there exists some $\lambda \in (0, 1)$ such that for every $n \ge 1$:

$$\left| \int \phi \cdot \psi \circ f^n d\mu - \int \phi d\mu \cdot \int \psi d\mu \right| \le C(\phi, \psi)\lambda^n,$$

for any Holder maps $\phi, \psi \in \mathcal{C}(M, \mathbb{R})$, where $C(\phi, \psi)$ depends only on the potentials ϕ, ψ.

Given now a Lebesgue space X and an endomorphism $f : X \to X$ preserving a probability measure μ, we say that $(X, f, \mathcal{B}(X), \mu)$ is an *exact endomorphism* ([8, 15, 16]) if

$$\bigcap_{n\geq 0} f^{-n}\mathcal{B}(X) = \mathcal{N},$$

where $\mathcal{N}$ is the σ-algebra containing only sets of μ-measure 0 or 1. Exact endomorphisms are important in ergodic theory and were first studied by Rohlin in [15, 16]. For instance he proved that $(X, f, \mathcal{B}(X), \mu)$ is exact if and only if for any measurable set of positive measure $A \subset X$, we have $\lim_{n\to\infty} \mu(f^n A) = 1$.

Now consider m-tuples of positive integers $\Delta = (k_1, \ldots, k_m)$ and denote by $\ell(\Delta) := \inf |k_i - k_j|, 1 \leq i < j \leq m$. We say that $(X, f, \mathcal{B}(X), \mu)$ is *mixing of order m* if for any arbitrary measurable sets $A_1, \ldots, A_m$ and for any sequences of m-tuples $\Delta^1 = (k_1^1, \ldots, k_m^1)$, $\Delta^2 = (k_1^2, \ldots, k_m^2), \ldots$, with $\lim_{n\to\infty} \ell(\Delta^n) = \infty$, we have

$$\lim_{n\to\infty} \mu\left(\bigcap_{i=1}^{m} f^{-k_i^n} A_i\right) = \prod_{i=1}^{m} \mu(A_i).$$

If $m = 2$ we obtain the usual notion of mixing measure. An exact endomorphism is mixing of any order, as shown in [16].

Theorem 5 *Let a repellor* Λ *for a smooth endomorphism as in Theorem 2 and let* μ^- *be the unique inverse SRB measure associated. Then*

(i) μ^- *has Exponential Decay of Correlations on Holder potentials*;
(ii) (Λ, μ^-) *is exact*, *and thus it is mixing of any order.*

Proof Since we have a uniformly hyperbolic structure for the endomorphism f on Λ, we can associate to it a Smale space structure on the natural extension $\hat{\Lambda}$ ([17]). Therefore on $\hat{\Lambda}$ there exist Markov partitions of arbitrarily small diameter ([17]). Now these Markov partitions imply the existence of a semi-conjugacy h with a 2-sided mixing Markov chain Σ_A. We have therefore the Lipschitz continuous maps

$$h : \Sigma_A \to \hat{\Lambda}, \quad \text{and} \quad \pi : \hat{\Lambda} \to \Lambda$$

such that $\pi \circ \hat{f} = f \circ \pi, h \circ \sigma_A = \hat{f} \circ h$, where σ_A is the shift homeomorphism.

Now, since the stable potential Φ^s on Λ is Holder continuous, it follows that $\Psi^s := \Phi^s \circ \pi \circ h : \Sigma_A \to \mathbb{R}$ is Holder continuous and to the unique equilibrium measure μ_s of Φ^s it corresponds the unique equilibrium measure ν of Ψ^s on Σ_A, s.t $\mu_s = (\pi \circ h)_* \nu$. We have that $P_f(\Phi^s) = P_{\sigma_A}(\Psi^s)$ and $h_{\mu_s}(f) = h_\nu(\sigma_A)$. Also notice that

$$\int_\Lambda \phi d\mu_s = \int_{\Sigma_A} \phi \circ \pi \circ h \, d\nu, \quad \phi \in \mathcal{C}(\Lambda).$$

Now we do have Exponential Decay of Correlations for Holder potentials for (Σ_A, ν) (for example [2]); so the same holds for $f|_\Lambda$ and the equilibrium measure μ_s. Recalling that we denoted $\mu^- := \mu_s$, we obtain (i).

For (ii) we shall use that $(\Lambda, f|_\Lambda, \mathcal{B}(\Lambda), \mu^-)$ is exact if and only if its natural extension is a K-automorphism ([16]). But its natural extension is $(\hat{\Lambda}, \hat{f}, \mathcal{B}(\hat{\Lambda}), \hat{\mu}_s)$, which is isomorphic to $(\Sigma_A, \sigma_A, \mathcal{B}(\Sigma_A), \nu)$ from the discussion above. We know that Markov chains with equilibrium measures of Holder potentials are K-automorphisms ([15]), hence the natural extension of (Λ, μ^-) is a K-automorphism. Thus $(\Lambda, f|_\Lambda, \mathcal{B}(\Lambda), \mu^-)$ is exact, and thus it is mixing of any order by the result in [16]. □

Examples

1. *Toral endomorphisms.* Let us take an integer valued $m \times m$ matrix A with $\det(A) \neq 1$. This matrix induces a toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m$. This toral endomorphism transforms the unit square into a parallelogram in $\mathbb{R}^m$ of area (Lebesgue measure) equal to $|\det(A)|$, and whose corners are points having only integer coordinates. Thus when we project to $\mathbb{T}^m$, we obtain that f_A is $|\det(A)|$-to-1. If all eigenvalues of A have absolute values different from 1, then f_A is hyperbolic on the whole torus $\mathbb{T}^m$.

Theorem 2 can be applied in this case, since $\mathbb{T}^m$ is a connected hyperbolic repellor for f_A, and we obtain a physical measure for the multivalued inverse iterates of f_A. In this case the inverse SRB measure μ^- is in fact the Haar measure on $\mathbb{T}^m$ since the stable potential is constant. Also from Theorem 3, we obtain that a Pesin type formula holds for the negative Lyapunov exponents.

2. *Anosov endomorphisms.* Theorems 2 and 3 can be applied also in the case of Anosov endomorphisms on a Riemannian manifold M, since M can be viewed as a hyperbolic repellor. In general the stable potential is not constant and μ^- is not necessarily absolutely continuous with respect to the Lebesgue measure on M. We obtain again that the asymptotic distribution of preimages for Lebesgue almost every point in M is equal to the equilibrium measure $\mu^- = \mu_s$, and that the inverse SRB measure μ^- has absolutely continuous conditional measures on local stable manifolds.

3. *Non-Anosov hyperbolic non-expanding repellors for products.* Let us take for instance $f : \mathbb{P}\mathbb{C}^1 \to \mathbb{P}\mathbb{C}^1$, $f([z_0 : z_1]) = [z_0^2 : z_1^2]$, and $g : \mathbb{T}^2 \to \mathbb{T}^2$, g being induced by the matrix $A = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 3 \end{smallmatrix}\right)$. We see easily that A has one eigenvalue in $(0, 1)$ and another larger than 1, so g is hyperbolic. We take the product

$$F : \mathbb{P}\mathbb{C}^1 \times \mathbb{T}^2 \to \mathbb{P}\mathbb{C}^1 \times \mathbb{T}^2, \quad F([z_0 : z_1], (x, y)) = (f([z_0 : z_1]), g(x, y)) \text{ and } \Lambda := S^1 \times \mathbb{T}^2.$$

Then Λ is a connected hyperbolic non-Anosov repellor for the smooth endomorphism F and we can apply Theorems 2 and 3.

4. *Perturbations.* According to Proposition 3, if f is hyperbolic on a connected repellor Λ and if an endomorphism g is a $\mathcal{C}^1$ perturbation of f, then g has a connected hyperbolic repellor denoted Λ_g which is close to Λ. We can form then a large class of examples by perturbing known examples, like the ones above. Then since g is hyperbolic on Λ_g we can again apply Theorems 2 and 3, this time for inverse SRB measures which might be more complicated than in the original (unperturbed) example. For instance, let us take $F : \mathbb{P}\mathbb{C}^1 \times \mathbb{T}^2 \to \mathbb{P}\mathbb{C}^1 \times \mathbb{T}^2$ given by

$$F([z_0 : z_1], (x, y)) = ([z_0^2 : z_1^2], f_A(x, y)),$$

where f_A is the toral endomorphism induced by the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix}\right)$. As can be seen, F has a connected hyperbolic repellor $\Lambda := S^1 \times \mathbb{T}^2$. Consider the following perturbation of F, $F_\varepsilon : \mathbb{P}\mathbb{C}^1 \times \mathbb{T}^2 \to \mathbb{P}\mathbb{C}^1 \times \mathbb{T}^2$ given by:

$$\begin{aligned} &F_\varepsilon([z_0 : z_1], (x, y)) \\ &\quad = \left([z_0^2 + \varepsilon z_1^2 \cdot e^{2\pi i(2x+y)} : z_1^2], (2x + y + \varepsilon \sin(2\pi(x + y)), 2x + 2y + \varepsilon \cos^2(4\pi x))\right). \end{aligned}$$

It can be seen that F_ε is well defined as a smooth endomorphism on $\mathbb{PC}^1 \times \mathbb{T}^2$ and that it is a $\mathcal{C}^1$ perturbation of F. It follows from Proposition 3 that F_ε has a connected hyperbolic repellor Λ_ε (on which F_ε has both stable as well as unstable directions), and that Λ_ε is close to Λ. However Λ_ε is different from Λ, and it has a complicated structure with self-intersections; its projection on the second coordinate is $\mathbb{T}^2$. For this repellor Λ_ε we can apply Theorem 2 to get a physical measure μ_ε^- for the local inverse iterates of F_ε. This physical measure μ_ε^- is the equilibrium measure of the non-constant stable potential

$$\Phi_\varepsilon^s([z_0 : z_1], (x, y)) := \log|\det(DF_\varepsilon)_s([z_0 : z_1], (x, y))|, \quad \text{for } ([z_0 : z_1], (x, y)) \in \Lambda_\varepsilon.$$

We know from Theorem 3 that the conditional measures of the inverse SRB measure μ_ε^- on the local stable manifolds (which are contained in the repellor Λ_ε), are absolutely continuous with respect to the induced Lebesgue measures.

Also a Pesin type formula is true for the measure-theoretic entropy $h_{\mu_\varepsilon^-}$ of μ_ε^-, and the negative Lyapunov exponents (which are non-constant if $\varepsilon \neq 0$).

Similarly one can perturb other connected hyperbolic repellors to obtain new dynamical systems for which Theorems 2 and 3, as well as Corollary 1 can be applied.

Another observation is that one can form repellors quite easily. We need only the existence of families of stable/unstable cones in some open set U and the topological condition $\bar{U} \subset f(U)$. Then one can form the basic set $\Lambda := \bigcap_{n\in\mathbb{Z}} f^n(U)$, on which we have a hyperbolic structure. The inverse SRB measure μ^- supported on Λ can be approximated Lebesgue almost everywhere on U, like in Theorem 2, and will have good ergodic properties as found in Theorem 5. However it may be difficult to describe this measure explicitly, especially in the non-Anosov case, since (Λ, μ^-) is not 1-sided Bernoulli.

Acknowledgements This work was partially supported by the PN II Project ID-1191 "Numerical invariants and geometric properties for classes of dynamical systems" from CNCSIS and the Romanian Ministry of Education and Research.

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